

# **Arbitrary replacement of a two-symbol entry of a given 4X4 diagonal MOLS and subsequent entry replacements**

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## **1. Introduction**

Mutually orthogonal Latin squares (hereafter MOLS in singular form) are known for centuries. In a MOLS, each entry in a square cell includes two symbols of respective two sets of symbols, each row and column include one and only one symbol of any of the two sets, and each symbol pair appears once and only once in the square. In the past, the term Graeco-Latin square was used as a Greek symbol set and a Latin symbol set were a popular choice. Usually, in a 4X4 MOLS the chosen set of Greek symbols includes  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$ , and the chosen set of Latin symbols includes A, B, C and D. In the following, a set of four two-symbol entries having each of the eight symbols once but only once is called an exclusive quartet.

In the current article I deal with 4X4 diagonal MOLS wherein besides each row and column being populated by an exclusive quartet, also each of the two diagonals populates exclusive quartet.

The current report presents a procedure which starts with a given 4X4 diagonal MOLS, relocates any two-symbol entry from any original square cell to any other destined square cell, and subsequently relocates all the other entries of the given MOLS in the other square cells without entry recalculation.

For that sake, I first demonstrate that for given symbol sets there are essentially only two MOLSs, that can be converted to each other by a plane rotation around a diagonal axis. For one of them, hereafter the generic MOLS, I show three operation families whose combination or partial combination being sufficient for transferring an entry location from an initial cell in the original MOLS to any other desired destined cell. Consequently, I outline a procedure to relocate a two-symbol entry located in any square cell of a given diagonal MOLS in any other square cell and rearrange the rest entries in the rest cells as well. Finally, I point out on possible application of the procedure in the field of recreational mathematics.

## **2. A generic MOLS**

Without loss of generality, Fig. 1a shows an exclusive quartet in a downward left to right diagonal of a 4X4 square. This choice of entries almost fully dictates the rest entries. It is left to the reader to verify that only a first and a second 4X4 diagonal MOLSs, shown in Fig. 1b and Fig. 1c, respectively, are compatible with the diagonal entries of Fig. 1a.

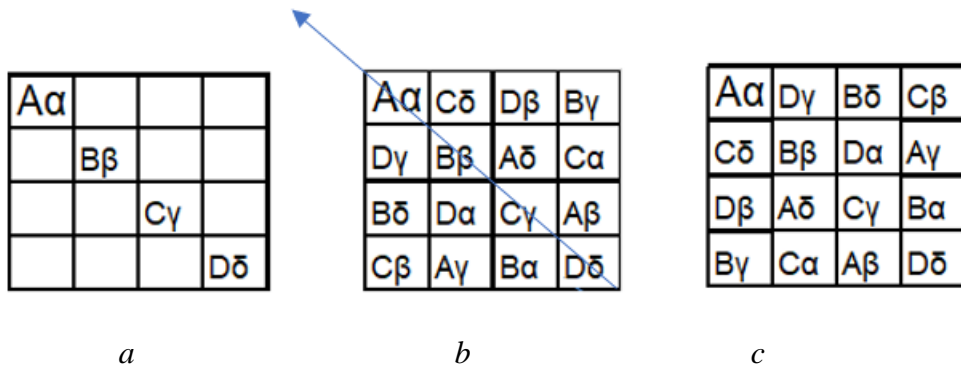


Fig. 1 A 4x4 diagonal MOLS. a. before calculation of entries outside the downward left to right diagonal. b. A first MOLS compatible with the MOLS of Fig. 1a. c. A second MOLS thereof.

It is noted that the first and second MOLS of respective Fig. 1b and Fig. 1c are essentially the same as 180° rotation of the first MOLS around a rotational axis along the downward left to right diagonal, as shown in Fig. 1b, results in the second MOLS. Thus, hereafter the first MOLS is called a generic 4X4 diagonal MOLS, and the discussion of the generic MOLS in the next section is valid for every 4X4 diagonal MOLS.

It is noted that besides the rows, columns and diagonals, there are several other exclusive quartets in the generic MOLS, including the four entries in the four corners, an exclusive quartet which populates the four internal cells, and the four quartets which populate respective corner quarters, hereafter the corner quartets.

In a corner quartet, the entries have a cyclic order. For example, going clockwise, the quartet of the first quarter has the entries A $\alpha$ , C $\delta$ , B $\beta$ , D $\gamma$ , in cyclic order. In other words, dealing with the quartet A $\alpha$ , C $\delta$ , B $\beta$ , D $\gamma$  is the same as dealing with the quartet C $\delta$ , B $\beta$ , D $\gamma$ , A $\alpha$ , for example.

### 3. Moving an entry within a square

In this section I discuss three operation families that relocate entries of the generic MOLS. It is shown that execution of certain operations of these families relocates any two-symbol entry from any original square cell to any destined cell.

The first operation family includes rotations of a whole MOLS around an axis which is normal to the page and penetrates the square in its center. Denoting the exclusive quartets of the four corner quarters of the generic MOLS by I, II, III, and IV, they are located initially in the first, second, third and fourth quarters, respectively, as shown in Fig. 2a. One may rotate the whole square clockwise by  $90^\circ$ ,  $180^\circ$ , and  $270^\circ$ , without affecting its orthogonality, as shown, for example, in Fig. 3 which presents a MOLS obtained after  $180^\circ$  rotation of the generic MOLS. The rotated locations of the four quartets are shown in Fig. 2b, 2c, and 2d, for the three rotations.

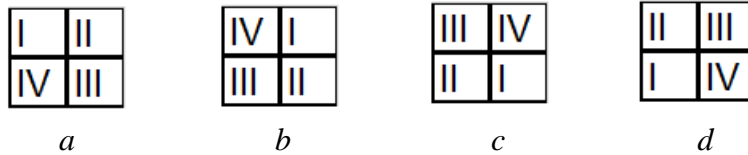


Fig. 2 The original location of the four exclusive corner quartets(a), after  $90^\circ$  rotation(b), after  $180^\circ$  rotation(c), and after  $270^\circ$  rotation(d).

|            |            |            |            |
|------------|------------|------------|------------|
| D $\delta$ | B $\alpha$ | A $\gamma$ | C $\beta$  |
| A $\beta$  | C $\gamma$ | D $\alpha$ | B $\delta$ |
| C $\alpha$ | A $\delta$ | B $\beta$  | D $\gamma$ |
| B $\gamma$ | D $\beta$  | C $\delta$ | A $\alpha$ |

Fig. 3 An orthogonal MOLS obtained by  $180^\circ$  rotation of the generic MOLS

Thus, a rotation of the first operation family relocates an entry from its original quarter to another quarter as desired, in dependence with an elected rotation angle selected from  $90^\circ$ ,  $180^\circ$ , and  $270^\circ$  rotations. Namely, in case that the original square cell and the destined square cell are in different quarters, an appropriate rotation of the whole MOLS relocates the quartet in the destined quarter corner.

The second operation family includes three rotations of a corner quartet around an axis which is normal to the page and penetrates the quarter in its center, thus relocating the entry in a desired cell within the quarter site. In the example of Fig. 4,

quartet I of the generic MOLS of Fig. 1b is rotated clockwise by  $90^\circ$ ,  $180^\circ$ , and  $270^\circ$ . Thus, for example,  $C\delta$  may be relocated at any other cell within the quarter.

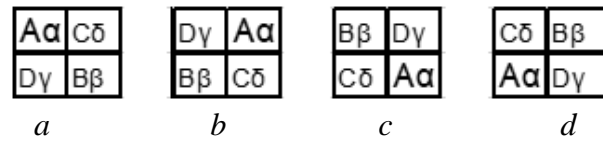


Fig. 4 The locations of the entries of quartet I (a), after  $90^\circ$  rotation(b), after  $180^\circ$  rotation(c), and after  $270^\circ$  rotation(d).

While the desired entry is relocated, the other quartet should be rotated by the same rotation to preserve the orthogonality of the whole MOLS. Fig. 5 shows the generic MOLS after  $180^\circ$  rotation of all the quartets I, II, III, and IV. The orthogonality is preserved despite the rotation of the quartets.

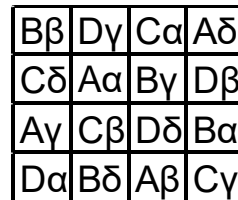


Fig. 5 A diagonal MOLS obtained by a  $180^\circ$  rotation of all the quartets.

In contrast, a  $90^\circ$  rotation of all the quartets results on a non-orthogonal square, as shown in Fig. 6a. Here enters the third operation family, cross-checking of quartet IV from the fourth quarter to the second quarter. Once cross-checked, the resulted square of Fig. 6b becomes orthogonal. It is left to the reader to verify that a  $270^\circ$  rotation of the quartets also necessitates quartet cross-checking.

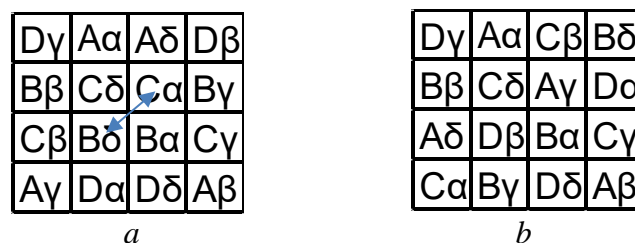


Fig. 6 A non-orthogonal square(a) obtained by  $90^\circ$  rotation of all the corner quartets converts to an orthogonal square(b) by cross-checking quartets II and IV.

The above discussion manifests that an entry of 4X4 diagonal MOLS can be relocated elsewhere at will and the rest entries may be relocated such as to preserve the orthogonality without a need to recalculate entries.

#### 4. A practical procedure to relocate entries in a 4X4 diagonal MOLS

Based on the discussion of section 3, I outline now a practical procedure to replace an entry from any desired original cell to any desired destined cell, and then rearrange all the other entries. In other words, starting with a 4x4 diagonal MOLS, the challenge is to relocate a two-symbol entry located initially at a certain original cell to a certain destined cell of a blank square and copy all the rest entries to the rest cells of the blank square. The procedure includes the following steps, which are explained by a way of an example.

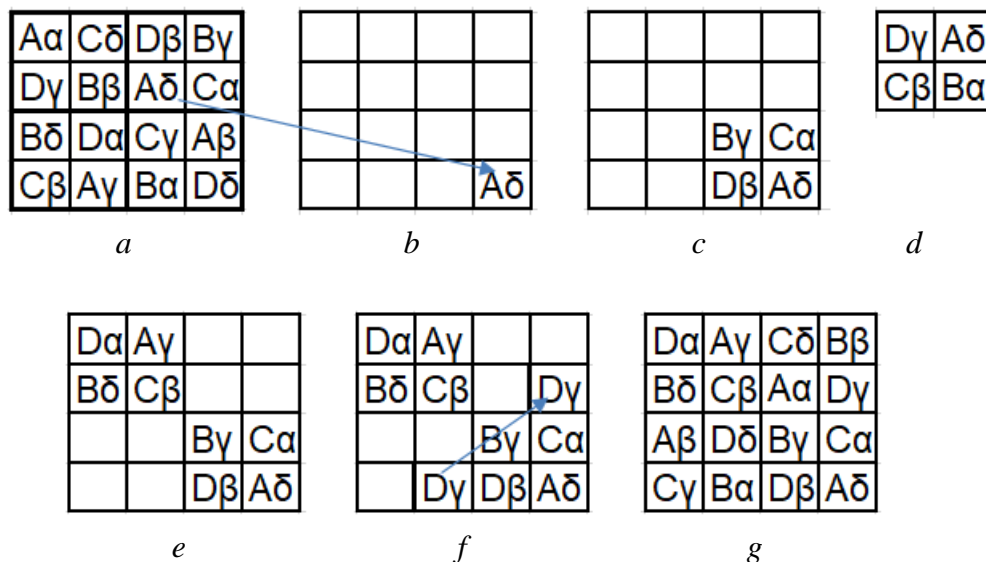


Fig. 7 A demonstration of a practical procedure to copy a two-symbol entry located initially at a cell, locate it in a destined cell of a blank square and copy all the rest entries to the rest cells of the blank square. a. the original square. b. the blank square with the desired entry at the destined cell. c. the other entries of quartet II copied to the right bottom quarter. d. four guiding entries. e. the members of quartet IV copied to the first quarter. f. a guiding entry of quartet I is found to be placed wrongly and delivered to the opposite quarter. g. the members of quartets I and III are copied to the destined square to provide an orthogonal 4X4 diagonal MOLS.

First, an arbitrary entry,  $A\delta$  of quartet II of the generic 4x4 diagonal MOLS of Fig. 7a, for example, is written down in the right bottom cell of a blank square, as shown in Fig. 7b. Next, the other entries of the quartet are written down, keeping their cyclic order in the quartet, as shown in Fig. 7c.

Fig. 7d shows four entries of quartets I, II, II, and IV that occupy the same relative cell in the original MOLS as  $A\delta$  of quartet II, the right bottom cell of the quarter.  $C\beta$  opposes  $A\delta$  in Fig. 7d. In the next step,  $C\beta$  is written down in the right bottom cell of the first quarter, relatively the same location that  $A\delta$  occupies in the third quarter. After that, the other entries of quartet IV are written down in the first quarter, again keeping their cyclic order in quartet IV, as shown in Fig. 7e.

In the next step,  $D\gamma$  of quartet I is initially written down in the right bottom cell of the fourth quarter. However, one notes that the lowest row has already  $D\beta$ , and thus  $D\gamma$  is not allowed to be there and is cross-checked to the right bottom cell of the second quarter, as shown in Fig. 7f. Again, the other entries of quartet I are written down, keeping their cyclic order in the quartet, as shown in Fig. 7g. Also,  $B\alpha$  of quartet III is written down in the right bottom cell of the third quarter, as well as the other entries of quartet III, which keep their place in the cyclic order. The resulted square of Fig. 7g is a 4X4 diagonal MOLS with the entry  $A\delta$  at the destined cell. No recalculation of entries was needed in that process, as desired.

## 5. A numerical Lando trick-an application in recreational mathematics

The numerical Lando trick uses the above discussion to perform a new version, hereafter the Lando trick, of a well known numerical trick. The term magic square is rigorously used for a square of order  $n$  which occupies the numbers  $1,2,\dots,n^2$  such that the sum of all entries of each row and column is the same. Also, a super magic square is a magic square where the sum of all diagonal terms is also the same as the rows and columns. It is well known, for centuries, that a magic square may be built using MOLS, as also detailed below. Anyhow, for the following application in recreational mathematics, the restriction of the number set to the number series  $1,2,\dots,n^2$  is removed.

Referring the generic 4x4 diagonal MOLS of Fig. 1b or 7a, the following list substitute numerical values for the Latin and Greek symbols as follows:

$$A=0; B=1; C=2; D=3$$

$$\alpha=0; \beta=4; \gamma=8; \delta=12.$$

Replacing the two substituted numbers of each entry by their sum, the super magic square of Fig. 8 is obtained. The same sum of 30 is obtained by rows, columns, diagonals, corner quartet, internal quartet, and corner quartets.

|    |    |    |    |
|----|----|----|----|
| 0  | 14 | 7  | 9  |
| 11 | 5  | 12 | 2  |
| 13 | 3  | 10 | 4  |
| 6  | 8  | 1  | 15 |

*Fig. 8 A super magic square obtained by substituting numbers for the Latin and Greek symbols of the generic MOLs and adding the numbers of each entry.*

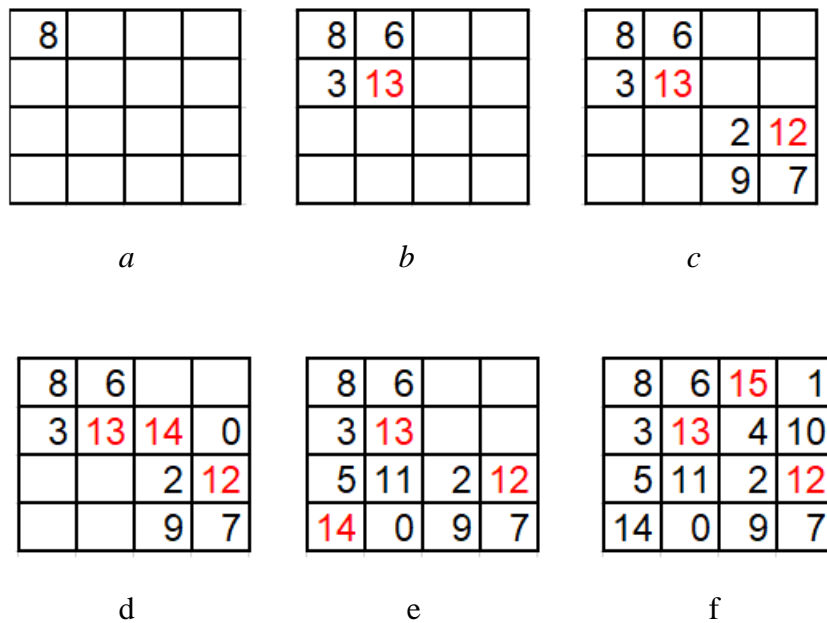
The entries which stem from substituting 12 for  $\delta$  are designated in red, and are dispersed once and only once in each row, column, diagonal, or quartet of the above list. In the well known number trick, a performer requests a participant to tell his/her age or guess a number. Consequently, the performer, who knows a super magic square by heart, adds or subtract a number to the red designated numbers and gets a sum which equals the guessed number, impressing the audience by the rapid calculation of the whole super magic square.

Referring now to Lando trick, it adds a challenge to the performer. The performer requests another participant to write down an arbitrary number from a set 0,1,2...11 in any of the cells of an empty square. The imposed number in a specific cell interferes the execution of the above trick. Instead, the performer must memorize by heart the following items:

- a. four cyclic series: 0,14,5,11; 7,9,2,12; 13,3,8,6; 10,4,15,1.
- b. 0 and 10 occupy the upper left cell of opposite quarters.
- c. 7 and 13 occupy the upper left cell of opposite quarters.

Fig. 9 demonstrates the trick with an example. First, by a chance, 8 is written down in the upper left cell of the first quarter. Then, the other numbers in the cyclic series of 8 are written down cyclically. As 13 have been written in the bottom right cell of the first quarter, 7 is written down in the bottom right cell of the third quarter, as well as the other entries of the cyclic series. Then, 0 is written down in the bottom right cell of the second quarter, as well as 14, the next number in the cyclic series. However, two red designated numbers 13 and 14 appear now in the same row which

is not allowed. Therefore, the 0,14 pair is cross-checked to the third quarter and the other numbers of the cyclic series are also written down. Finally, 10 is written down in the bottom right cell of the second quarter, as well as the other entries of the cyclic series.



*Fig. 9 An example of Lando trick. a. 8 is written down in the upper left cell of the first quarter. b. The other numbers in the cyclic series of 8 are written down cyclically. c. 7 is written down in the bottom right cell of the third quarter, as well as the other entries of the cyclic series. d. 0 is written down in the bottom right cell of the second quarter, as well as 14, the next number in the cyclic series. e. the 0,14 pair is cross-checked to the third quarter and the other numbers of the cyclic series are written down. f. 10 is written down in the bottom right cell of the second quarter, as well as the other entries in the cyclic series.*

To complete the trick, a number is subtracted or added to the four red designated numbers to get a desired sum, as occurs in the well known trick.

## 6. Conclusions

The discussion showed that certain operations enable to rearrange entries of a 4X4 diagonal MOLS such that one entry is relocated at will and there is no need to recalculate the other entries, but merely to rearrange them according to certain rules. Consequently, a Lando trick is demonstrated as a challenging version of a well known



number trick. The reader is challenged to find out an application to the described procedure in the field of statistics, which is also known to use MOLS.